Homework 5 answer

2021 年 11 月 19、26 日布置

2021 年 12 月 3 日交

1

对于球面

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

计算其 Gauss 曲率：

$$\begin{aligned}K = &\frac{1}{2g}\left[2\frac{\partial^2 g_{12}}{\partial x_1 \partial x_2} - \frac{\partial^2 g_{11}}{\partial x_2^2} - \frac{\partial^2 g_{22}}{\partial x_1^2}\right] \\ &- \frac{g_{22}}{4g^2}\left[\left(\frac{\partial g_{11}}{\partial x_1}\right)\left(2\frac{\partial g_{12}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1}\right) - \left(\frac{\partial g_{11}}{\partial x_2}\right)^2\right] \\ &+ \frac{g_{12}}{4g^2}\left[\left(\frac{\partial g_{11}}{\partial x_1}\right)\left(\frac{\partial g_{22}}{\partial x_2}\right) - 2\left(\frac{\partial g_{11}}{\partial x_2}\right)\left(\frac{\partial g_{22}}{\partial x_1}\right) + \left(2\frac{\partial g_{12}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2}\right)\left(2\frac{\partial g_{12}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1}\right)\right] \\ &- \frac{g_{11}}{4g^2}\left[\left(\frac{\partial g_{22}}{\partial x_2}\right)\left(2\frac{\partial g_{12}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2}\right) - \left(\frac{\partial g_{22}}{\partial x_1}\right)^2\right]\end{aligned}$$

答案为 $K = \frac{1}{a^2}$。解答时请把每一项写出来

行列式 $g = a^4 \sin^2\theta$

$$\begin{aligned}K =&\frac{1}{2a^4\sin^2\theta}\left[-\frac{\partial^2}{\partial\theta^2}(a^2\sin^2\theta)\right] \\ &- 0 \\ &+ 0 \\ &- \frac{a^2}{4a^8\sin^4\theta}\left[-\left(\frac{\partial}{\partial\theta}(a^2\sin^2\theta)\right)^2\right] \\ =&\frac{1}{a^2\sin^2\theta}(\cos^2\theta - \cos 2\theta) \\ =&\frac{1}{a^2}\end{aligned}$$

2

局域 Minkowski 坐标系（自由落体坐标系）$\{\xi^\mu\}$ 中 Christoffel symbol ($\Gamma^\lambda{}_{\mu\nu}$) 为 0，在另外一任意坐标系 $\{x^\mu\}$ 中, 证明 $\{x^\mu\}$ 系中的克氏符为

$$\Gamma^\lambda{}_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$$

$\{x^\mu\}$ 系中度规为:

$$g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}$$

逆变形式为:

$$g^{\mu\nu} = \eta^{\alpha\beta}\frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial x^\nu}{\partial \xi^\beta}$$

Christoffel symbol 为 (其中蓝色部分和红色部分分别相消)

$$\begin{aligned}
\Gamma^\lambda{}_{\mu\nu} =& \frac{1}{2}g^{\lambda\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})\\
=& \frac{1}{2}\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial x^\sigma}{\partial \xi^\beta}\eta^{\alpha\beta}\left[\frac{\partial}{\partial x^\nu}\left(\frac{\partial \xi^\gamma}{\partial x^\mu}\frac{\partial \xi^\rho}{\partial x^\sigma}\eta_{\gamma\rho}\right) + \frac{\partial}{\partial x^\mu}\left(\frac{\partial \xi^\gamma}{\partial x^\nu}\frac{\partial \xi^\rho}{\partial x^\sigma}\eta_{\gamma\rho}\right) - \frac{\partial}{\partial x^\sigma}\left(\frac{\partial \xi^\gamma}{\partial x^\mu}\frac{\partial \xi^\rho}{\partial x^\nu}\eta_{\gamma\rho}\right)\right]\\
=& \frac{1}{2}\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial x^\sigma}{\partial \xi^\beta}\eta^{\alpha\beta}\left[\frac{\partial^2 \xi^\gamma}{\partial x^\mu \partial x^\nu}\frac{\partial \xi^\rho}{\partial x^\sigma}\eta_{\gamma\rho} + \frac{\partial \xi^\gamma}{\partial x^\mu}\frac{\partial^2 \xi^\rho}{\partial x^\nu \partial x^\sigma}\eta_{\gamma\rho}\right.\\
&+ \frac{\partial^2 \xi^\gamma}{\partial x^\mu \partial x^\nu}\frac{\partial \xi^\rho}{\partial x^\sigma}\eta_{\gamma\rho} + \frac{\partial \xi^\gamma}{\partial x^\nu}\frac{\partial^2 \xi^\rho}{\partial x^\mu \partial x^\sigma}\eta_{\gamma\rho}\\
&\left. - \underbrace{\frac{\partial^2 \xi^\gamma}{\partial x^\mu \partial x^\sigma}\frac{\partial \xi^\rho}{\partial x^\nu}\eta_{\gamma\rho}}_{symmetric\ over\ \gamma,\rho} - \frac{\partial \xi^\gamma}{\partial x^\mu}\frac{\partial^2 \xi^\rho}{\partial x^\nu \partial x^\sigma}\eta_{\gamma\rho}\right]\\
=& \frac{1}{2}\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial x^\sigma}{\partial \xi^\beta}\eta^{\alpha\beta}\left[2\frac{\partial^2 \xi^\gamma}{\partial x^\mu \partial x^\nu}\frac{\partial \xi^\rho}{\partial x^\sigma}\eta_{\gamma\rho}\right]\\
=& \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\gamma}{\partial x^\mu \partial x^\nu}\underbrace{\frac{\partial x^\sigma}{\partial \xi^\beta}\frac{\partial \xi^\rho}{\partial x^\sigma}}_{\delta^\rho{}_\beta}\eta^{\alpha\beta}\eta_{\gamma\rho}\\
=& \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\gamma}{\partial x^\mu \partial x^\nu}\underbrace{\eta^{\alpha\beta}\eta_{\gamma\beta}}_{\delta^\alpha{}_\gamma}\\
=& \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}
\end{aligned}$$

有很多同学犯了这样的错误:

$$\begin{aligned}
\Gamma^\lambda{}_{\mu\nu} =& \frac{1}{2}g^{\lambda\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma})\\
=& \frac{1}{2}\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial x^\sigma}{\partial \xi^\beta}\eta^{\alpha\beta}\left[\frac{\partial}{\partial x^\nu}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\sigma}\eta_{\alpha\beta}\right) + \cdots\right]
\end{aligned}$$

这样的话一项中就出现了 4 个 α，β，这是 illegal 的，Einstein 求和约定每一项只能有两个指标相同，超过两个指标会出现混乱。更何况这里分别是不同的 2 组求和，在同一项中不能混用相同的指标。犯了这

样错的同学基本上同时还错上加错:

$$\eta^{\alpha\beta}\eta_{\alpha\beta} = 1$$

这显然是不对的，正确的结果是：

$$\eta^{\alpha\beta}\eta_{\alpha\beta} = \delta^{\alpha}{}_{\alpha} = \delta^{0}{}_{0} + \delta^{1}{}_{1} + \delta^{2}{}_{2} + \delta^{3}{}_{3} = 1 + 1 + 1 + 1 = 4$$