双星 GW 辐射

Q

$A^\mu // U^\mu \Rightarrow A^\mu_{//} = A^\mu$

$A^\mu = \lambda U^\mu \Rightarrow A^\mu_{//} = \Pi^\mu{}_\nu A^\nu = -U^\mu U_\nu \lambda U^\nu$

$U^\mu U_\mu \quad U^\mu$

$A^\mu B_\mu \; C^\mu D_\mu$

HW 11

1. 在 $w^\mu_{\hat\alpha}$ 中展开： $\dot{w}^\mu_{\hat\alpha} = \Omega_{\hat\alpha\hat\beta} w^{\hat\beta\mu}$ $\quad w_{\hat\gamma\mu}$ 证明 $\Omega_{\hat\alpha\hat\beta} = -\Omega_{\hat\beta\hat\alpha}$

正交归一： $w^\mu_{\hat\alpha} w^\nu_{\hat\beta} g_{\mu\nu} = \eta_{\hat\alpha\hat\beta}$

完备性： $w^\mu_{\hat\alpha} w^{\hat\alpha}_\nu = \delta^\mu{}_\nu$

$\eta^{\hat\beta}{}_{\hat\alpha}$

右 $= \Omega_{\hat\alpha\hat\beta} w^{\hat\beta\mu} w_{\hat\gamma\mu} = \Omega_{\hat\alpha\hat\beta}\delta^{\hat\beta}{}_{\hat\gamma} = \Omega_{\hat\alpha\hat\gamma}$

$$\text{左} = \dot{w}^\mu_{\hat\alpha} w_{\hat\beta\mu} = \underbrace{\frac{D}{D\tau}(\underbrace{w^\mu_{\hat\alpha} w^\nu_{\hat\gamma} g_{\mu\nu}}_{\eta_{\hat\alpha\hat\gamma}})}_{0} - w^\mu_{\hat\alpha}\dot{w}^\nu_{\hat\beta} g_{\mu\nu} = -w^\mu_{\hat\alpha}\Omega_{\hat\beta\hat\gamma} w^{\hat\gamma\nu} g_{\mu\nu}$$

$$= -\Omega_{\hat\beta\hat\alpha}$$

2. Ω

HW 12

标准形式：$ds^2 = -B\,dt^2 + A\,dr^2 + r^2 d\Omega_2$

1. static spherical symmetric 度规的谐和坐标形式

$$\Box^2 t = 0$$
$$\Box^2 X^i = 0$$

$$\begin{cases} X^1 = R(r)\sin\theta\cos\varphi \\ X^2 = R(r)\sin\theta\sin\varphi \\ X^3 = R(r)\cos\theta \end{cases}$$

① $\Gamma^\lambda_{\mu\nu}$

$$\Box^2 X^i = g_{\mu\nu}\nabla^\mu\nabla^\nu X^i = g^{\mu\nu}(X^i{}_{,\mu,\nu} - \underline{\Gamma^\lambda_{\mu\nu}} X^i{}_{,\lambda})$$

$$\Box^2 t = g^{\mu\nu}(\underbrace{t_{,\mu,\nu}}_{0} - \Gamma^\theta_{\mu\nu}\underbrace{t_{,\lambda}}_{0,\ t}) = \underline{-g^{\mu\nu}\Gamma^t_{\mu\nu}} = 0$$

$$\Box^2 X^i = g^{\mu\nu}(X^i{}_{,\mu,\nu} - \Gamma^\lambda_{\mu\nu} X^i{}_{,\lambda})$$

$$= g^{rr}X^i{}_{,r,r} + g^{\theta\theta}X^i{}_{,\theta,\theta} + g^{\varphi\varphi}X^i{}_{,\varphi,\varphi} - \sum_\mu g^{\mu\mu}\underline{\Gamma^r_{\mu\mu}}X^i{}_{,r} - g^{\varphi\varphi}\Gamma^\theta_{\varphi\varphi}X^i{}_{,\theta}$$

$$= \frac{1}{A}\frac{R''}{R}X^i - \frac{1}{r^2}X^i + \frac{1}{r^2\sin^2\theta}X^i{}_{,\varphi,\varphi} + \left(-\frac{A'}{2A^2} + \frac{2}{Ar} + \frac{B'}{2AB}\right)\frac{R'}{R}X^i + \frac{\cot\theta}{r^2}X^i{}_{,\theta}$$

$$= \frac{X^i}{AR}\left[R'' + \left(-\frac{A'}{2A^2} + \frac{2}{Ar} + \frac{B'}{2AB}\right)R' - \frac{A}{r^2}R\right] + \underline{\frac{1}{r^2\sin^2\theta}X^i{}_{,\varphi,\varphi} + \frac{\cot\theta}{r^2}X^i{}_{,\theta}}$$

$i = 1,2$ $\quad = \frac{1}{r^2\sin^2\theta}(-X^i) + \frac{\cot\theta}{r^2}\cot\theta\, X^i = \underline{-\frac{1}{r^2}X^i}$

$i = 3$ $\quad = 0 + \frac{\cot\theta}{r^2}(-\tan\theta)X^i = \underline{-\frac{1}{r^2}X^i}$

$$\underline{\Box^2 X^i} = \frac{X^i}{AR}\left[R'' + (\qquad\qquad)R' - \frac{2A}{r^2}R\right]$$

$$\underline{\frac{d}{dr}(r^2 B^{1/2} A^{-1/2} R') - 2A^{1/2}B^{1/2}R = 0}$$

② ③

$$\Box^2\varphi = (g^{\mu\nu}\varphi_{;\nu})_{;\mu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\, g^{\mu\nu}\varphi_{,\nu})_{,\mu}$$

$$g_{tt} = -B \quad g_{rr} = A$$

$$\boxed{g = -ABr}$$

$$\Box^2 t = \frac{1}{\sqrt{-g}}(\sqrt{-g}\, g^{tt} t_{,t})_{,t} = 0$$

$$X^i = R(r)\,\Theta^i(\theta,\varphi)$$

$$\Box^2 X^i = \frac{1}{\sqrt{-g}}\left\{(\sqrt{-g}\, g^{rr} X^i_{,r})_{,r} + (\sqrt{-g}\, g^{\theta\theta} X^i_{,\theta})_{,\theta} + (\sqrt{-g}\, g^{\varphi\varphi} X^i_{,\varphi})_{,\varphi}\right\}$$

$$= \frac{1}{\sqrt{-g}}\left\{\frac{\partial}{\partial r}\left(r^2 A^{-1/2} B^{1/2}\sin\theta\,\frac{\partial X^i}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(A^{1/2}B^{1/2}\sin\theta\,\frac{\partial X^i}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}A^{1/2}B^{1/2}\frac{\partial X^i}{\partial\varphi}\right)\right\}$$

$$= \frac{1}{\sqrt{-g}}\left\{\frac{d}{dr}(r^2 A^{-1/2}B^{1/2}R')\sin\theta\,\Theta^i + A^{1/2}B^{1/2}R\sin\theta\underbrace{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\Theta^i) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\Theta^i\right]}_{-L^2\Theta^i}\right\}$$

$$Y_1^m$$

$$-l(l+1)\Theta^i$$
$$\|$$
$$-2\Theta^i$$

$l=1$

$$Y_1^1 \propto -\sin\theta\, e^{i\varphi}$$

$$Y_1^0 \propto \cos\theta$$

$$Y_1^{-1} \propto \sin\theta\, e^{-i\varphi}$$

$$Y_1^{-1} + Y_1^1 \propto \sin\theta\sin\varphi = \Theta^1$$

$$Y_1^0 \propto \Theta^3$$

$$Y_1^{-1} - Y_1^1 \propto \Theta^2$$

2. 史瓦西解的各向同性形式

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega_2$$

$$-\left(1-\frac{2GM}{r}\right)dt^2 + J(\rho)\left(d\rho^2 + \rho^2 d\Omega_2\right)$$

$$\begin{cases} J(\rho)\,d\rho^2 = \left(1-\frac{2GM}{r}\right)^{-1}dr^2 \\ \rho^2 J(\rho) = r^2 \end{cases}$$

$$\frac{d\rho}{\rho} = \frac{dr}{\sqrt{r^2-2GMr}} \Rightarrow r = \rho\left(1+\frac{GM}{2\rho}\right)^2$$

3. 史瓦西的谐和坐标形式

$$\frac{d}{dr}(r^2 B^{1/2}A^{-1/2}R') - 2A^{1/2}B^{1/2}R = 0 \quad\Rightarrow\quad R = r - GM$$

4. GR下的等效势能

① geodesic eqns 推导出来 ② 求解 $ds^2=-Bdt^2+Adr^2+r^2d\Omega$

推导 测地线方程

① Killing 矢量

度规不含 t， $K^\mu_{(t)}=\partial_t=(1,0,0,0)$

$$-E=K^\mu P_\mu=P_0=g_{0\mu}P^\mu=g_{0\mu}\frac{dx^\mu}{d\lambda}=g_{00}\dot t=-B\dot t$$

度规不含 φ， $K^\mu_{(\varphi)}=\partial_\varphi=(0,0,0,1)$

$$L=K^\mu P_\mu=P_3=g_{33}\dot\varphi=r^2\sin^2\theta\,\dot\varphi=r^2\dot\varphi$$

$$g_{\mu\nu}\dot x^\mu\dot x^\nu=-\epsilon=\begin{cases}1 & \text{massive}\\ 0 & \text{massless}\end{cases}$$

$$\Rightarrow B\dot t^2-A\dot r^2-r^2\dot\varphi^2=\epsilon \Longrightarrow \frac{E^2}{B}-A\dot r^2-\frac{L^2}{r^2}=\epsilon$$

$\theta=\frac{\pi}{2}$

$$\begin{cases}B\dot t=E\\ r^2\dot\varphi=L\\ \theta=\frac{\pi}{2}\\ A\dot r^2+\frac{L^2}{r^2}-\frac{E^2}{B}=-\epsilon\end{cases}$$

仿射参数是 τ

② Lagrange 法 相对论天体物理 书

$$\boxed{L=\frac{1}{2}g_{\mu\nu}\dot x^\mu\dot x^\nu}$$

以史瓦西为例 $L\approx-\frac{1}{2}(1-\frac{2GM}{r})+\frac{1}{2}(1+\frac{2GM}{r})(\frac{dr}{dt})^2+\frac{1}{2}r^2(\frac{d\theta}{dt})^2+\frac{1}{2}r^2s^2\theta(\frac{d\varphi}{dt})^2$

$\frac{2GM}{r}\ll1$, $t=\tau$ $\quad=-\frac{1}{2}-(-\frac{GM}{r})+\frac{1}{2}(\frac{dr}{dt})^2=T-V$

E−L 方程 $\boxed{\frac{d}{d\lambda}(\frac{\partial L}{\partial\dot x^\mu})-\frac{\partial L}{\partial x^\mu}=0}$

$$\begin{cases}\frac{\partial L}{\partial\dot t}=g_{tt}\dot t=-B\dot t=-E\\ \frac{\partial L}{\partial\dot\varphi}=r^2\dot\varphi=L\end{cases}$$

$$H=P_\mu\dot x^\mu-L=\frac{\partial L}{\partial\dot x^\mu}\dot x^\mu-L=2L-L=L=-\frac{1}{2}\epsilon \Rightarrow g_{\mu\nu}\dot x^\mu\dot x^\nu=-\epsilon$$

$\theta=\frac{\pi}{2}$

③ $\Gamma^\lambda{}_{\mu\nu}$

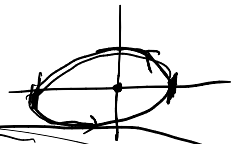
Weinberg

$$\begin{cases}\boxed{B\dot t=1}\\ r^2\dot\varphi=J\\ \theta=\frac{\pi}{2}\\ A\dot r^2+\frac{J^2}{r^2}-\frac{1}{B}=-E\end{cases}$$

仿射参数 p

$$\begin{cases} B\dot t = 1 \\ r^2\dot\varphi = J \\ \theta = \frac{\pi}{2} \\ A\dot r^2 + \frac{J^2}{r^2} - \frac{1}{B} = -E \end{cases}$$

$r(\varphi)$



$$\frac{A}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2 B} = -\frac{E}{J^2} \Longrightarrow \varphi = \mp\int \frac{A^{1/2}\,dr}{r^2\left[\frac{1}{J^2B} - \frac{E}{J^2} - \frac{1}{r^2}\right]^{1/2}}$$

轨道进动

$$\frac{dr}{d\varphi} = 0 \Longrightarrow \boxed{\frac{1}{r^2} - \frac{1}{J^2B} = -\frac{E}{J^2}} \qquad r_-,\ r_+$$

$$\begin{cases} E = \dfrac{r_+^2/B_+ - r_-^2/B_-}{r_+^2 - r_-^2} \\[2ex] J^2 = \dfrac{1/B_+ - 1/B_-}{1/r_+^2 - 1/r_-^2} \end{cases}$$

$$\frac{1}{J^2B} - \frac{E}{J^2} - \frac{1}{r^2} = \frac{r_-^2(B^{-1} - B_-^{-1}) - r_+^2(B^{-1} - B_+^{-1})}{r_+^2 r_-^2 (B_+^{-1} - B_-^{-1})} - \frac{1}{r^2}$$

$$\frac{1}{J^2B} - \frac{E}{J^2} - \frac{1}{r^2} = C\left(\frac{1}{r_-} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{r_+}\right) \qquad r\to\infty \qquad C$$

$$B = 1 - \frac{2GM}{r} + 2(\beta-\gamma)\frac{G^2M^2}{r^2} \Longrightarrow B^{-1} \approx 1 + 2\frac{GM}{r} + 2(2-\beta+\gamma)\frac{G^2M^2}{r^2}$$

$$r\to\infty:\quad -C\frac{1}{r_- r_+} = \frac{r_-^2(1 - B_-^{-1}) - r_+^2(1 - B_+^{-1})}{r_+^2 r_-^2(B_+^{-1} - B_-^{-1})}$$

$$C = \frac{r_+^2\left(-\frac{2GM}{r_+} - 2(2-\beta+\gamma)\frac{G^2M^2}{r_+^2}\right) - r_-^2\left(-\frac{2GM}{r_-} - 2(2-\beta+\gamma)\frac{G^2M^2}{r_-^2}\right)}{r_+ r_-\left[2GM\left(\frac{1}{r_+} - \frac{1}{r_-}\right) + 2(2-\beta-\gamma)G^2M^2\left(\frac{1}{r_+^2} - \frac{1}{r_-^2}\right)\right]}$$

$$= \frac{-2GM(r_+ - r_-)}{2GM(r_- - r_+)\left[1 + 2(2-\beta+\gamma)GM\left(\frac{1}{r_+} + \frac{1}{r_-}\right)\right]}$$

$$\approx 1 - (2-\beta+\gamma)GM\left(\frac{1}{r_+} + \frac{1}{r_-}\right) + O\left(\frac{GM}{r_\pm}\right)^2$$

$$A = 1 + 2\gamma\frac{GM}{r} \Longrightarrow \sqrt{A} = 1 + \gamma\frac{GM}{r}$$

$$\varphi(r) - \varphi(r_-) = C^{-1/2}\int_{r_-}^{r} \frac{1 + \gamma\frac{GM}{r}}{r^2\sqrt{(r_-^{-1} - r^{-1})(r^{-1} - r_+^{-1})}}\,dr$$

$$\frac{1}{r} \equiv \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) + \frac{1}{2}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)\sin\psi \qquad \begin{aligned} r_-&: \psi = -\tfrac{\pi}{2} \\ r_+&: \psi = \tfrac{\pi}{2} \end{aligned}$$

$$\varphi(r) - \varphi(r_-) = C^{-1/2}\int \frac{\left\{1 + \gamma\frac{GM}{2}\left[(r_+^{-1} + r_-^{-1}) + (r_+^{-1} - r_-^{-1})\sin\psi\right]\right\}}{\frac{1}{2}(r_-^{-1} - r_+^{-1})\sqrt{1 - \sin^2\psi}}\,\frac{1}{2}\left(\frac{1}{r_-} - \frac{1}{r_+}\right)d(\sin\psi)$$

$$= C^{-1/2}\int_{-\frac{\pi}{2}}^{\psi} \frac{1 + \gamma\frac{GM}{2}\left[(r_+^{-1} + r_-^{-1}) + (r_+^{-1} - r_-^{-1})\sin\psi\right]}{\cancel{\cos\psi}}\,\cancel{\cos\psi}\,d\psi$$

$$= \left[1 + \frac{1}{2}(2-\beta+2\gamma)GM\left(\frac{1}{r_+} + \frac{1}{r_-}\right)\right]\left(\psi + \frac{\pi}{2}\right) - \gamma\frac{GM}{2}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)\cos\psi$$

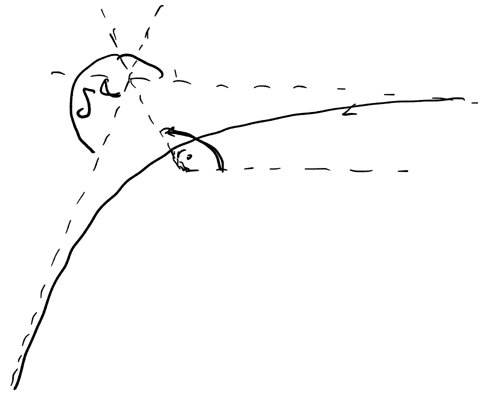
$$\varphi(r_+) - \varphi(r_-) = \pi + \frac{\pi}{2}(2-\beta+2\gamma)GM\left(\frac{1}{r_+} + \frac{1}{r_-}\right)$$

$$\Delta\varphi = 2|\varphi(r_+) - \varphi(r_-)| - 2\pi = \frac{2-\beta+2\gamma}{3}\,\frac{6\pi GM}{L} \qquad \frac{1}{L} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right)$$

光线偏折

$$\varphi = \pm\int \frac{A^{1/2}\,dr}{r^2\sqrt{\frac{1}{J^2B}-\frac{E}{J^2}-\frac{1}{r^2}}} = \pm\int \frac{A^{1/2}\,dr}{r^2\sqrt{\frac{1}{J^2B}-\frac{1}{r^2}}}$$

$\frac{dr}{d\varphi}$



$$2|\varphi(r_0)-\varphi(\infty)|-\pi = \delta$$

$$J^2 = \frac{r_0^2}{B_0}$$

$$\varphi(r)-\varphi(\infty) = -\int_\infty^{r_0} \frac{A^{1/2}\,dr}{r^2\sqrt{\frac{B_0}{B}\frac{r^2}{r_0^2}-1}}$$

$$B = 1-2\frac{GM}{r}$$

$$\Rightarrow \left(\frac{r}{r_0}\right)^2\frac{B_0}{B}-1 \approx \left(\frac{r}{r_0}\right)^2\left(1-2\frac{GM}{r_0}+\frac{2GM}{r}\right)-1 = \left[\left(\frac{r}{r_0}\right)^2-1\right]\left[1-\frac{2GMr}{r_0(r+r_0)}\right]$$

$$A = 1+2\gamma\frac{GM}{r} \Rightarrow \sqrt{A} = 1+\gamma\frac{GM}{r}$$

$$\varphi(r)-\varphi(\infty) = \int_{r_0}^{\infty}\frac{1}{r\sqrt{r^2/r_0^2-1}}\left(1+\gamma\frac{GM}{r}+\frac{GMr}{r_0(r+r_0)}\right)dr$$

$$= \int_{r_0}^{\infty}\left\{\frac{r_0}{r\sqrt{r^2-r_0^2}} + \gamma GM\frac{r_0}{r^2\sqrt{r^2-r_0^2}} + GM\frac{1}{(r+r_0)\sqrt{r^2-r_0^2}}\right\}dr$$

$$= \arctan\sqrt{\left(\frac{r}{r_0}\right)^2-1}\,\Big|_r^{\infty} + \gamma GM\sqrt{\frac{1}{r_0^2}-\frac{1}{r^2}}\,\Big|_r^{\infty} + \frac{GM}{r_0}\sqrt{\frac{r-r_0}{r+r_0}}\,\Big|_r^{\infty}$$

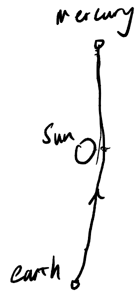
$$= \frac{\pi}{2} - \arctan\sqrt{\left(\frac{r}{r_0}\right)^2-1} + (\gamma+1)\frac{GM}{r_0} - \gamma GM\sqrt{\frac{1}{r_0^2}-\frac{1}{r^2}} - \frac{GM}{r_0}\sqrt{\frac{r-r_0}{r+r_0}}$$

$$\varphi(r_0)-\varphi(\infty) = \frac{\pi}{2}+(\gamma+1)\frac{GM}{r_0}$$

$$\delta = 2\left(\frac{\pi}{2}+(\gamma+1)\frac{GM}{r_0}\right)-\pi = \frac{4GM}{r_0}\left(\frac{1+\gamma}{2}\right)$$

雷达回波延迟

$$\frac{A}{B^2}\left(\frac{dr}{dt}\right)^2+\frac{J^2}{r^2}-\frac{1}{B} = 0$$



$$J^2 = \frac{r_0^2}{B_0}$$

$$\frac{A}{B}\left(\frac{dr}{dt}\right)^2+\left(\frac{r_0}{r}\right)^2\frac{1}{B_0}-\frac{1}{B} = 0 \Rightarrow \frac{dt}{dr} = \pm\sqrt{\frac{A/B}{1-\frac{B}{B_0}\left(\frac{r_0}{r}\right)^2}}$$

从 r 到 r_0 的时间 $t(r,r_0) = \int_{r_0}^{r}\sqrt{\frac{A/B}{1-\frac{B}{B_0}\left(\frac{r_0}{r}\right)^2}}\,dr$

$$B = 1-\frac{2GM}{r} \Rightarrow 1-\frac{B}{B_0}\left(\frac{r_0}{r}\right)^2 \approx \left[1-\left(\frac{r_0}{r}\right)^2\right]\left[1-\frac{2GMr_0}{r(r+r_0)}\right]$$

$$\sqrt{A/B} \approx 1+(1+\gamma)\frac{GM}{r}$$

$$t(r,r_0) = \int_{r_0}^{r}\frac{1}{\sqrt{1-r_0^2/r^2}}\left(1+(1+\gamma)\frac{GM}{r}+\frac{GMr_0}{r(r+r_0)}\right)dr$$

$$= \int_{r_0}^{r}\left\{\frac{r}{\sqrt{r^2-r_0^2}} + (1+\gamma)GM\frac{1}{\sqrt{r^2-r_0^2}} + GM\frac{r_0}{(r+r_0)\sqrt{r^2-r_0^2}}\right\}dr$$

$$= \sqrt{r^2-r_0^2}\,\Big|_{r_0}^{r} + (1+\gamma)GM\ln\left(r+\sqrt{r^2-r_0^2}\right)\Big|_{r_0}^{r} + GM\sqrt{\frac{r-r_0}{r+r_0}}\,\Big|_{r_0}^{r}$$

$$t(r, r_0) = \sqrt{r^2 - r_0^2} + (1+\gamma) GM \ln\left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\right) + GM\sqrt{\frac{r - r_0}{r + r_0}}$$

$r_0 = R_\odot \ll r_E, r_M$

$$\Delta t = 2\left[t(r_E, R_\odot) + t(r_M, R_\odot) - \sqrt{r_E^2 - R_\odot^2} - \sqrt{r_M^2 - R_\odot^2}\right]$$

$$= 2\left\{(1+\gamma) GM \ln\left(\frac{r_E + \sqrt{r_E^2 - R_\odot^2}}{R_\odot}\right) + GM\sqrt{\frac{r_E - R_\odot}{r_E + R_\odot}} + (r_E \to r_M)\right\}$$

$$\approx 2\left\{(1+\gamma) GM \ln \frac{2r_E}{R_\odot} + GM + (r_E \to r_M)\right\}$$

$$= 4GM\left\{1 + \frac{1+\gamma}{2} \ln \frac{4 r_E r_M}{R_\odot^2}\right\}$$

黑洞阴影、 引力透镜

波
